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# EXPRESSIONS OF EULER ANGLES AS FUNCTIONS OF THE DIRECTIONAL COSINES OF THREE-ORTHOGONAL AXES IN RIGID BODY KINEMATICS - I

BY

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**Abstract.** One of the main problems of kinematics is the reference of the mobile frame to the fixed frame. The position of the mobile frame (position of the body) in relation to the fixed frame is given by the orthogonal base-change matrix (also known as rotation matrix, or the matrix of the directional cosines) whose elements are expressed with the Euler angles.

This work establishes reverse link relations between Euler angles and direction cosines, relations that are equivalent to the known formulae in the field literature.

**Keywords:** three-orthogonal axes; Euler angles; directional cosines.

## 1. Introduction

There is considered a three-orthogonal fixed frame *OXYZ* with the unit vectors of the axes marked with $\bar{i}$, $\bar{j}$, $\bar{k}$, and a three-orthogonal mobile frame $QX'Y'Z'$ with the unit vectors of the axes denoted as $\bar{i}'$, $\bar{j}'$, $\bar{k}'$. It is supposed that the frames have the same origin, $O \equiv Q$, that is, only the rotational motion is taken into consideration in order to define the Euler angles (Fig. 1).

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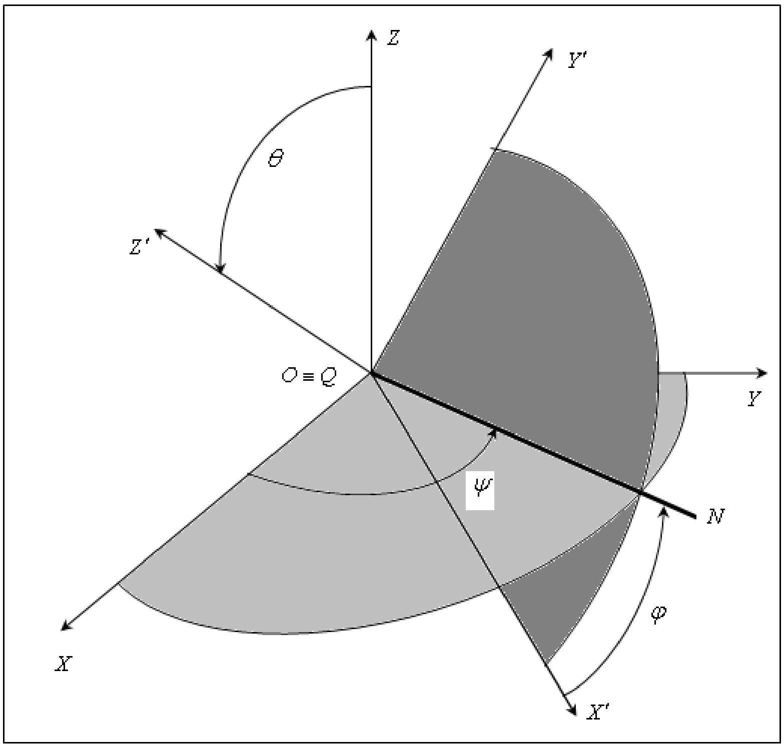


Fig. 1 – The Euler angles.

Between the unit vectors of the mobile axes and the unit vectors of the fixed axes there is the following relation (Condurache and Burlacu, 2014; Ibănescu and Rusu, 1998; Mangeron and Irimiciuc, 1978):

$$\begin{bmatrix} \bar{i}' \\ \bar{j}' \\ \bar{k}' \end{bmatrix} = A \begin{bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{bmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{1}$$

where $A$ is the base-change matrix (the matrix of the directional cosines). Matrix $A$ is an orthogonal and invertible matrix ( $A \cdot A^T = I_3$, $I_3$ being the unit matrix of order 3, (Condurache and Burlacu, 2014; Ibănescu and Rusu, 1998; Mangeron and Irimiciuc, 1978). The inverse matrix of $A$ is obtained by transposition; therefore, Eq. (1) can be written in the equivalent form

$$\begin{bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{bmatrix} = A^{-1} \begin{bmatrix} \bar{i}' \\ \bar{j}' \\ \bar{k}' \end{bmatrix}, \quad A^{-1} = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \tag{2}$$

The elements of matrix $A$ satisfy the conditions for the definition of the unit vectors, that are the unitary norm and orthogonality conditions.

The conditions for the unit vectors of the mobile frame are:

$$\begin{aligned} a_{11}^2 + a_{12}^2 + a_{13}^2 = 1 \\ a_{21}^2 + a_{22}^2 + a_{23}^2 = 1 \\ a_{31}^2 + a_{32}^2 + a_{33}^2 = 1 \end{aligned} \tag{3}$$

$$\begin{aligned} a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = 0 \\ a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0 \\ a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} = 0 \end{aligned} \tag{4}$$

The conditions for the unit vectors of the fixed frame are:

$$\begin{aligned} a_{11}^2 + a_{21}^2 + a_{31}^2 = 1 \\ a_{12}^2 + a_{22}^2 + a_{32}^2 = 1 \\ a_{13}^2 + a_{23}^2 + a_{33}^2 = 1 \end{aligned} \tag{5}$$

$$\begin{aligned} a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0 \\ a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0 \\ a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = 0 \end{aligned} \tag{6}$$

## 2. Content

The number of the independent elements of the matrix $A$ is equal to the number of rotational freedom degrees expressed within the three Euler angles (Ibănescu and Rusu, 1998; Mangeron and Irimiciuc, 1978; Tocaci, 1985). Of the nine elements of matrix $A$ only three are independent, that is, if three elements of the matrix are known (not all three on the same line or on the same column), the other six elements result from the equations system that expresses conditions for the definition of the unit vectors, namely the conditions given by Eq. (3) to Eq.(6).

From these twelve conditions, only six are independent (Ibănescu and Rusu, 1998; Mangeron and Irimiciuc, 1978), for example, conditions given by Eq. (3) with Eq. (4) or conditions given by Eq. (5) with Eq. (6).

The Euler angles are position angles between the mobile frame $OX'Y'Z'$ and the fixed frame $OXYZ$. These angles are defined as follows (Ibănescu and Rusu, 1998; Mangeron and Irimiciuc, 1978), being represented in Fig. 1.

There is defined the node line as the intersection line between plane $OXY$ and plane $OX'Y'Z'$:

$$(N) = OXY \bigcap OX'Y' \tag{7}$$

There is defined the nutation angle as the angle between axes $OZ$ and $OZ'$:

$$\sphericalangle\theta = \sphericalangle(OZ, OZ'), \quad \sphericalangle\theta \in [0^\circ, 180^\circ] \tag{8}$$

There is defined the unit vector of the node line by the following vector operation:

$$\bar{n} = \frac{\bar{k} \times \bar{k}'}{\sin\theta} \quad , \quad \sphericalangle\theta \neq 0^\circ, 180^\circ \tag{9}$$

There is defined the precession angle as the angle between the fixed axis $OX$ and the unit vector $\vec{n}$, measured directly in the plane $OXY$:

$$\sphericalangle\psi = \sphericalangle(OX, ON), \quad \sphericalangle\psi \in [0^\circ, 360^\circ) \tag{10}$$

There is defined the own rotation angle as the angle between the mobile axis $OX'$ and the unit vector $\vec{n}$, measured directly in the plane $OX'Y'$:

$$\sphericalangle\varphi = \sphericalangle(OX', ON), \quad \sphericalangle\varphi \in [0^\circ, 360^\circ) \tag{11}$$

The definition relations of the angles $\psi$ and $\varphi$ are the following:

$$\bar{i} \times \bar{n} = \bar{k}\sin\psi, \quad \bar{i}' \times \bar{n} = \bar{k}'\sin\varphi \tag{12}$$

The elements of the matrix of the directional cosines are expressed in the literature by the Euler angles (Ibănescu and Rusu, 1998; Mangeron and Irimiciuc, 1978) by the following relationships:

$$\begin{aligned}
&a_{11} = \bar{i}' \cdot \bar{i} = \cos\varphi \cdot \cos\psi + \sin\varphi \cdot \sin\psi \cdot \cos\theta \\
&a_{12} = \bar{i}' \cdot \bar{j} = \cos\varphi \cdot \sin\psi - \sin\varphi \cdot \cos\psi \cdot \cos\theta \\
&a_{13} = \bar{i}' \cdot \bar{k} = -\sin\varphi \cdot \sin\theta \\
&a_{21} = \bar{j}' \cdot \bar{i} = \sin\varphi \cdot \cos\psi - \cos\varphi \cdot \sin\psi \cdot \cos\theta \\
&a_{22} = \bar{j}' \cdot \bar{j} = \sin\varphi \cdot \sin\psi + \cos\varphi \cdot \cos\psi \cdot \cos\theta \\
&a_{23} = \bar{j}' \cdot \bar{k} = \cos\varphi \cdot \sin\theta \\
&a_{31} = k' \cdot \bar{i} = \sin\psi \cdot \sin\theta \\
&a_{32} = \bar{k}' \cdot \bar{j} = -\cos\psi \cdot \sin\theta \\
&a_{33} = \bar{k}' \cdot \bar{k} = \cos\theta
\end{aligned} \tag{13}$$

For the unit vector of the node line, there are written the relations (Ibănescu and Rusu, 1998; Mangeron and Irimiciuc, 1978)

$$\bar{n} = \bar{i} \cdot \cos\psi + \bar{j} \cdot \sin\psi = \bar{i}' \cdot \cos\varphi + \bar{j}' \cdot \sin\varphi \tag{14}$$

Replacing the expressions of the mobile unit vectors $\bar{i}'$ and $\bar{j}'$ according to Eq. (1), it results

$$\begin{aligned}
\bar{i} \cdot \cos\psi + \bar{j} \cdot \sin\psi = (\bar{i} \cdot a_{11} + \bar{j} \cdot a_{12} + \bar{k} \cdot a_{13}) \cdot \cos\varphi + \\
+ (\bar{i} \cdot a_{21} + \bar{j} \cdot a_{22} + \bar{k} \cdot a_{23}) \cdot \sin\varphi
\end{aligned} \tag{15}$$

from which, it results that

$$\begin{aligned}
&a_{11} \cdot \cos\varphi + a_{21} \cdot \sin\varphi = \cos\psi \\
&a_{12} \cdot \cos\varphi + a_{22} \cdot \sin\varphi = \sin\psi \\
&a_{13} \cdot \cos\varphi + a_{23} \cdot \sin\varphi = 0
\end{aligned} \tag{16}$$

The last relationship in Eq. (16) has the following solutions:

$$\cos\varphi = \pm \frac{a_{23}}{\sqrt{{a_{13}}^2 + a_{23}^2}}, \quad \sin\varphi = \mp \frac{a_{13}}{\sqrt{a_{13}^2 + a_{23}^2}} \tag{17}$$

By replacing Eq. (17) in the first relationship from Eq. (16) it results that

$$\cos\psi = \frac{\pm a_{11} \cdot a_{23} \mp a_{21} \cdot a_{13}}{\sqrt{a_{13}^2 + a_{23}^2}} \tag{18}$$

By replacing Eq. (17) in the second relationship from Eq. (16) it results that

$$\sin\psi = \frac{\pm a_{12} \cdot a_{23} \mp a_{22} \cdot a_{13}}{\sqrt{a_{13}^2 + a_{23}^2}} \tag{19}$$

The Eq. (18) and the Eq. (19) verify the following equality

$$\cos^2\psi + \sin^2\psi = 1 \tag{20}$$

According to the definition of the unit vectors, there applies the following

$$\overline{i'} \times \overline{j'} = \overline{k'} \tag{21}$$

According to Eq. (1), the first two unit vectors of the mobile frame are written as follows

$$\begin{aligned} \overline{i'} &= a_{11} \cdot \overline{i} + a_{12} \cdot \overline{j} + a_{13} \cdot \overline{k} \\ \overline{j'} &= a_{21} \cdot \overline{i} + a_{22} \cdot \overline{j} + a_{23} \cdot \overline{k} \end{aligned} \tag{22}$$

By replacing Eq. (22) in Eq. (21), there is obtained the following equality

$$(a_{12}a_{23} - a_{22}a_{13})\overline{i} - (a_{11}a_{23} - a_{21}a_{13})\overline{j} + (a_{11}a_{22} - a_{21}a_{12})\overline{k} = \overline{k'} \tag{23}$$

By the scalar multiplication of Eq. (23) with the unit vector $\overline{k}$ it results

$$a_{11}a_{22} - a_{21}a_{12} = \overline{k'} \cdot \overline{k} = \cos\theta \ , \tag{24}$$

where the definition of the scalar product and that of the nutation angle were considered.

By the scalar multiplication of Eq. (23) with the unit vectors $\bar{i}$ and $\bar{j}$ it results

$$\begin{aligned} a_{12}a_{23} - a_{22}a_{13} = \bar{k}' \cdot \bar{i} \\ a_{21}a_{13} - a_{11}a_{23} = \bar{k}' \cdot \bar{j} \end{aligned} \tag{25}$$

If the expression of the unit vector $\bar{k}'$ according to the Euler angles (Ibănescu and Rusu, 1998; Mangeron and Irimiciuc, 1978)

$$\bar{k}' = \bar{i} \cdot \sin\psi \cdot \sin\theta - \bar{j} \cdot \cos\psi \cdot \sin\theta + \bar{k} \cdot \cos\theta \tag{26}$$

is scalar multiplied by $\bar{i}$ and $\bar{j}$, it results

$$\bar{k}' \cdot \bar{i} = \sin\psi \cdot \sin\theta, \qquad \bar{k}' \cdot \bar{j} = -\cos\psi \cdot \sin\theta \tag{27}$$

From the Eq. (25) and Eq. (27) it results that

$$\begin{aligned} a_{12}a_{23} - a_{22}a_{13} = \sin\psi \cdot \sin\theta \\ a_{11}a_{23} - a_{21}a_{13} = \cos\psi \cdot \sin\theta \end{aligned} \tag{28}$$

According to Eq. (1), Eq. (2) and Eq. (24) it results

$$\bar{k}' \cdot \bar{k} = \cos\theta = a_{33} \tag{29}$$

For the nutation angle, there is written that

$$\sin\theta = \sqrt{1 - \cos^2\theta} \quad , \quad \sin\theta \geq 0 \tag{30}$$

According to Eq. (29) and Eq. (30), there applies that

$$\sin\theta = \sqrt{1 - a_{33}^2} \tag{31}$$

By replacing Eq. (31) in the third relationship from Eq. (5) it results:

$$\sin\theta = \sqrt{a_{13}^2 + a_{23}^2} \tag{32}$$

By replacing Eq. (18), Eq. (19) and Eq. (32) in Eq. (28), there are obtained the following identities

$$\begin{aligned} a_{12}a_{23} - a_{22}a_{13} &= \pm a_{12}a_{23} \mp a_{22}a_{13} \\ a_{11}a_{23} - a_{21}a_{13} &= \pm a_{11}a_{23} \mp a_{21}a_{13} \end{aligned} \tag{33}$$

from which there results that, in the relations that give the angles $\varphi$ and $\psi$ (Eq. (17), Eq. (18) and Eq. (19)) there is considered the upper sign.

## 3. Conclusions

In conclusion, the relations that express the Euler angles according to the direction cosines, that is the reversed relations of Eq. (13), are Eq. (17), Eq. (18), Eq. (19) and Eq. (24).

The precession angle is obtained by using Eq. (18) and Eq. (19):

$$\cos\psi = \frac{a_{11}a_{23} - a_{21}a_{13}}{\sqrt{a_{13}^2 + a_{23}^2}}, \quad \sin\psi = \frac{a_{12}a_{23} - a_{22}a_{13}}{\sqrt{a_{13}^2 + a_{23}^2}} \tag{34}$$

For the own rotation angle, are used the relationships in Eq. (17):

$$\cos\varphi = \frac{a_{23}}{\sqrt{a_{13}^2 + a_{23}^2}}, \quad \sin\varphi = -\frac{a_{13}}{\sqrt{a_{13}^2 + a_{23}^2}} \tag{35}$$

For the nutation angle Eq. (24) becomes:

$$\cos\theta = a_{11}a_{22} - a_{21}a_{12}. \tag{36}$$

The definition domains of the Euler angles are the following:

$$\sphericalangle\psi \in [0^\circ, 360^\circ), \quad \sphericalangle\varphi \in [0^\circ, 360^\circ), \quad \sphericalangle\theta \in [0^\circ, 180^\circ].$$

The relations given by Eq. (34), Eq. (35) and Eq. (36) contain the following six directional cosines:

$$a_{11},\ a_{12},\ a_{13},\ a_{21},\ a_{22},\ a_{23}$$

from which three are independents.

Observing that the terms in Eq. (34) are obtained by the development of the vector product Eq. (21) with the expression of the unit vectors in Eq. (1) and according to Eq. (29), Eq. (31) and Eq. (32), the Eq. (34) to Eq. (36) are written with five directional cosines as:

$$\begin{aligned}&\cos\psi = -\frac{a_{32}}{\sqrt{1-a_{33}^2}}, \quad \sin\psi = \frac{a_{31}}{\sqrt{1-a_{33}^2}}\\ &\cos\varphi = \frac{a_{23}}{\sqrt{1-a_{33}^2}}, \quad \sin\varphi = -\frac{a_{13}}{\sqrt{1-a_{33}^2}}\\ &\cos\theta = a_{33}\end{aligned} \tag{37}$$

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## EXPRESIILE UNGHIURILOR LUI EULER ÎN FUNCȚIE DE COSINUSURILE DIRECTOARE ALE AXELOR REPERULUI TRI-ORTOGONAL ÎN CINEMATICA RIGIDULUI - I


(Rezumat)

Una dintre problemele de bază ale cinematicii este raportarea reperului mobil la reperul fix. Poziţia reperului mobil în raport cu reperul fix este dată de matricea de schimbare a bazei (matricea de rotaţie sau matricea cosinusurilor directoare) ale cărei elemente sunt definite cu unghiurile lui Euler. În lucrare se stabilesc relaţii de legătură inversă între unghiurile lui Euler şi cosinusurile directoare, relaţii echivalente cu formulele din literatura de specialitate.